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LETTER TO THE EDITOR

Localized solutions with chaotic and fractal behaviours in a (2 + 1)-dimensional dispersive long-wave system

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Abstract
It is known that in high dimensions there exist abundant localized excitations such as dromions, lumps, ring soliton solutions and so on. In this paper, the possible chaotic and fractal localized structures are revealed for the (2 + 1)-dimensional dispersive long-wave equation. The chaotic and fractal dromion and lump patterns of the model are constructed by some types of lower-dimensional chaotic and fractal patterns.

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Since the concept of the fractal was proposed by Mandelbrot in 1975, it has seen the most remarkable developments in mathematical science and has had important influences on biology and social science. The fractal has also been widely applied to physics such as fractal growth phenomena including diffusion-limited aggregation, viscous fingering, cracks, etc, and the physical properties (such as diffusion, flow, vibration magnetism, etc) of fractal structures such as percolation clusters, polymers, porous media and so on. Usually, solitons, chaos and fractals are the most important three parts of nonlinear science [1]. Conventionally, these three parts are treated independently. In other words, one does not discuss the possibility of the existence of chaos and fractals in a soliton systems.

However, in a recent study of soliton systems, we have found that some lower-dimensional arbitrary functions can be included in the exact solutions of some (2 + 1)-dimensional integrable models. Now the important and interesting question is what will happen when these lower-dimensional arbitrary functions are taken as solutions of some lower-dimensional chaotic systems or some types of fractal functions. In particular, are there some types of high-dimensional localized excitations with some kinds of chaotic and/or fractal behaviours?

To answer these questions concretely, we take the (2+1)-dimensional dispersive long-wave system (DLWS) as a simple example, which reads

$$\begin{aligned} &u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \\ &v_t + (uv + u + u_{xy})_x = 0. \end{aligned} \tag{1}$$

The (2 + 1)-dimensional DLWS (1) is one of the integrable extensions [2, 3] of the (1 + 1)-dimensional integrable DLWS,

$$\begin{aligned} &u_t + v_x + uu_x = 0, \\ &v_t + (uv + u + u_{xx})_x = 0 \end{aligned} \tag{2}$$

which are derived in the context of a water wave propagating in narrow infinitely long channels of finite constant depth [4]. Some interesting properties of the system (1), which were first obtained by Boiti *et al* [5], have been studied by several authors [5–8]. For instance, the model (1) is Lax and IST (inverse scattering transformation) integrable [5] but not Painlevé integrable [6]. In [7] and [8], it was pointed out that the (2 + 1)-dimensional DLWE possesses Kac–Moody–Virasoro type and general w_∞ symmetry algebras with some arbitrary functions, which shows us that the exact solution of the model may have some arbitrary functions.

To solve the (2 + 1)-dimensional DLWE (1), we take the transformation

$$u = 2(\ln f)_x + u_0(x, t), \qquad v = 2(\ln f)_{xy} - 1 \tag{3}$$

which can be obtained from the standard Painlevé truncation expansion with $u = u_0(x, t)$, an arbitrary function of $\{x, t\}$, and $v = -1$ a seed solution of (1).

Substituting (3) into (1) yields two multi-linear equations in f which degenerate to the same trilinear form:

$$\begin{aligned} 2((f_{xyt} + u_{1x} f_{xy} + u_1 f_{xxy}) + f_{xxxy}) f^2 + 2(-f_x f_y u_{1x} - (u_1 f_{xx} + f_{xt} + f_{xxx}) f_y \\ - (f_{xxy} + 2u_1 f_{xy} + f_{yt}) f_x - f_{xy} f_{xx} - f_t f_{xy}) f \\ + 4 f_x f_y (u_1 f_x + f_{xx} + f_t) = 0. \end{aligned} \tag{4}$$

To find some interesting special solutions of (4), we can use the variable separation ansatz

$$f = 1 + a_1 p(x, t) + a_2 q(y, t) + a_3 p(x, t) q(y, t), \tag{5}$$

where a_1, a_2, a_3 are arbitrary constants and $p \equiv p(x, t)$ and $q = q(y, t)$ are functions of the indicated variables only. Substituting equation (5) into (4) yields

$$\begin{aligned} [2(a_1 + a_3 q) - (a_0 + a_1 p + a_2 q + a_3 pq) p_x^{-1} \partial_x](p_t + p_{xx} + u_1 p_x) \\ + [2(a_2 + a_3 p) - (a_0 + a_1 p + a_2 q + a_3 pq) q_y^{-1} \partial_y] q_t = 0. \end{aligned} \tag{6}$$

Because p is y-independent and q is x-independent, equation (6) can be separated into two equations:

$$p_t = -p_{xx} - p_x u_0 - (a_1 a_2 - a_3)(c_2 p^2 - c_1 p + c_0), \tag{7}$$

$$q_t = (a_1 + a_3 q)^2 c_0 + (1 + a_2 q)^2 c_2 + (a_1 + a_3 q)(1 + a_2 q) c_1, \tag{8}$$

where $c_i \equiv c_i(t)$, $i = 1, 2, 3$ are arbitrary functions of t.

Although it is not an easy task to obtain general solutions of equations (7) and (8) for any fixed u_0, we can treat the problem in an alternative way. Because both u_0 and p are functions of $\{x, t\}$, we can view p as an arbitrary function while fixing u_0 by equation (7). As to equation (8), its general solution has the form

$$q = \frac{A_1}{A_3 + F(y)} + A_2 \tag{9}$$

with $F(y)$ being an arbitrary function of y, while the new arbitrary functions A_1, A_2 and A_3 are linked with c_1, c_2 and c_3 by

$$\begin{aligned} c_2 &= -\frac{a_3(A_2 a_3 + a_1) A_{1t}}{(a_1 a_2 - a_3)^2 A_1} + \frac{a_3^2 A_{2t}}{(a_1 a_2 - a_3)^2} - \frac{(A_2 a_3 + a_1)^2 A_{3t}}{(a_1 a_2 - a_3)^2 A_1} \\ c_1 &= \frac{(a_1 a_2 + 2 A_2 a_2 a_3 + a_3) A_{1t}}{(a_1 a_2 - a_3)^2 A_1} - \frac{2 a_2 a_3 A_{2t}}{(a_1 a_2 - a_3)^2} + \frac{2(A_2 a_3 + a_1)(a_2 A_2 + 1) A_{3t}}{(a_1 a_2 - a_3)^2 A_1} \\ c_0 &= -\frac{a_2(a_2 A_2 + 1) A_{1t}}{(a_1 a_2 - a_3)^2 A_1(t)} + \frac{a_2^2 A_{2t}}{(a_1 a_2 - a_3)^2} - \frac{(a_2 A_2 + 1)^2 A_{3t}}{(a_1 a_2 - a_3)^2 A_1}. \end{aligned} \tag{10}$$

Finally, substituting (5) into (3), we obtain a quite general solution of the (2 + 1)-dimensional DLWE (1)

$$u = \frac{2(a_1 + a_3 q)p_x}{1 + a_1 p + a_2 q + a_3 pq} + u_0 \tag{11}$$

$$v1 \equiv v + 1 = -\frac{2(a_1 a_2 - a_3)p_x q_y}{(1 + a_1 p + a_2 q + a_3 pq)^2} \tag{12}$$

where p is an arbitrary function of $\{x, t\}$, q is determined by equation (9) and u_0 is determined by equation (7) with (10).

It is worth pointing out that the expression for the physical quantity $v1$ (12) is valid for some other physical models such as the asymmetric Nizhnik–Novikov–Veselov (ANNV) equation [9] and the asymmetric Darvey–Stewartson (ADS) equation. If q in (12) is taken as an arbitrary function of $\{y, t\}$, then the expression (12) is also valid for some types of quantities of the NNV and DS systems [10].

As pointed out in [9–11], because p and $F(y)$ are arbitrary functions, there exist many stable localized soliton solutions such as the multi-dromion solutions, multi-lump solutions and multi-ring soliton solutions. Furthermore, if the functions p and q are selected as some types of chaotic functions then some types of chaotic non-localized patterns the solutions of (12) such as the chaotic–chaotic patterns, periodic–chaotic patterns and the chaotic line soliton patterns can be found [12]. Now the most important question is whether we can find some types of chaotic localized solutions (say, the chaotic dromions and lumps) for the high-dimensional soliton system. Actually, the answer is obviously positive because the functions p, $F(y)$, $A_1(t)$, $A_2(t)$ and $A_3(t)$ are arbitrary functions. There are many types of possible chaotic and fractal dromion and lump patterns because any types of chaos and fractal models can be used to construct localized solutions. In the remainder of this letter, we appropriately select the arbitrary functions to reveal some special types of chaotic and fractal dromions and lump solutions for the field $v1$ (12) of the 2DLWE with $a_1 = 1, a_2 = 1, a_3 = 2$, i.e.

$$v1 = 2\frac{p_x q_y}{(1 + p + q + 2pq)^2}. \tag{13}$$

(i) Chaotic dromions. Setting p and q as

$$p = (50 + h(t))(1 + e^x), \qquad q = e^y, \tag{14}$$

with

$$A_1 - 1 = A_2 = A_3 = c_0 = c_1 = c_2 = 0, \qquad q = \frac{1}{F(y)}, \tag{15}$$

where $h(t)$ is an arbitrary function of t. From the expression (13) with (14), we know that the amplitude of the dromion is determined by the function $h(t)$. In particular, if we select the function $h(t)$ as a solution of a chaotic system, for instance, the well known Lorenz system [13]

$$f_t = -10(f - g), \qquad g_t = f(60 - h) - g, \qquad h_t = fg - \tfrac{8}{3}h, \tag{16}$$

then the field $v1$ becomes a typical chaotic dromion solution. In the selection (14), only one function h is included explicitly. The functions f and g are used to determine the function h via (16) (one may also write down a single complicated third-order ordinary differential equation for the function h by ruling out the functions f and g from (14)). At any fixed value of $h(t)$, we can get a single-dromion solution as shown in figure 1(*a*) (at a fixed time given by $h(t) = 0$). The amplitude of the dromion is changed chaotically with $h(t)$ as depicted in figure 1(*b*).

(ii) Regular fractal dromions and lumps with self-similar structures. Recently, some types of piecewise smooth solutions such as the peakons, cuspons and compactons have been

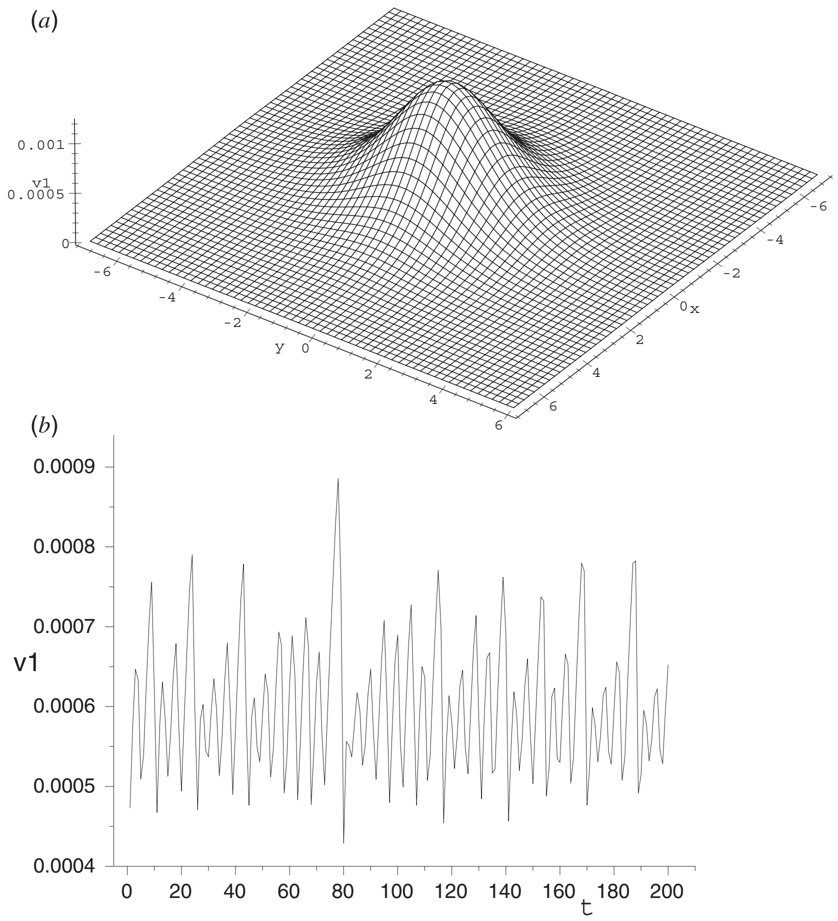

Figure 1. Plot of the single solution for the field $v1$ given by (13) with (14). (*a*) The full structure of the dromion when $h(t) = 0$. (*b*) The evolution of the amplitude of the chaotic dromion.

widely applied in $(1+1)$-dimensional soliton systems [14]. All the smooth lower-dimensional piecewise functions can be used to construct higher-dimensional peakons, cuspons and compactons. Furthermore, to our surprise and excitement, many lower-dimensional piecewise smooth functions with fractal structures can be used to construct exact localized solutions of higher-dimensional soliton systems which also possess fractal structures. In (2+1)-dimensions, one type of the most important basic excitations are so-called dromions which are exponentially localized in all directions. Selecting the arbitrary functions of (13) appropriately, we may obtain some kinds of fractal dromions. One of the simple fractal dromions (13) with (15) and

$$
\begin{aligned}
p &= \exp(-x(x + \sin(\ln(x^2)) - \cos(\ln(x^2)))), \\
q &= \exp(-y(y + \sin(\ln(y^2)) - \cos(\ln(y^2)))),
\end{aligned}
\tag{17}
$$

plotted in figures 2(*a*), (*b*) is a density plot of the fractal structure of the dromion at the region $\{x = [-0.07, 0.07], y = [-0.032, 0.032]\}$. To observe the self-similar structure of the fractal dromion more clearly, one may enlarge a small region near the centre of figure 2(*b*). For instance, if we reduce the region of figure 2(*b*) to $\{x = [-0.0066, 0.0066], y = [-0.0066, 0.0066]\}$, $\{x = [-0.001\,38, 0.001\,38], y = [-0.001\,38, 0.001\,38]\}$,

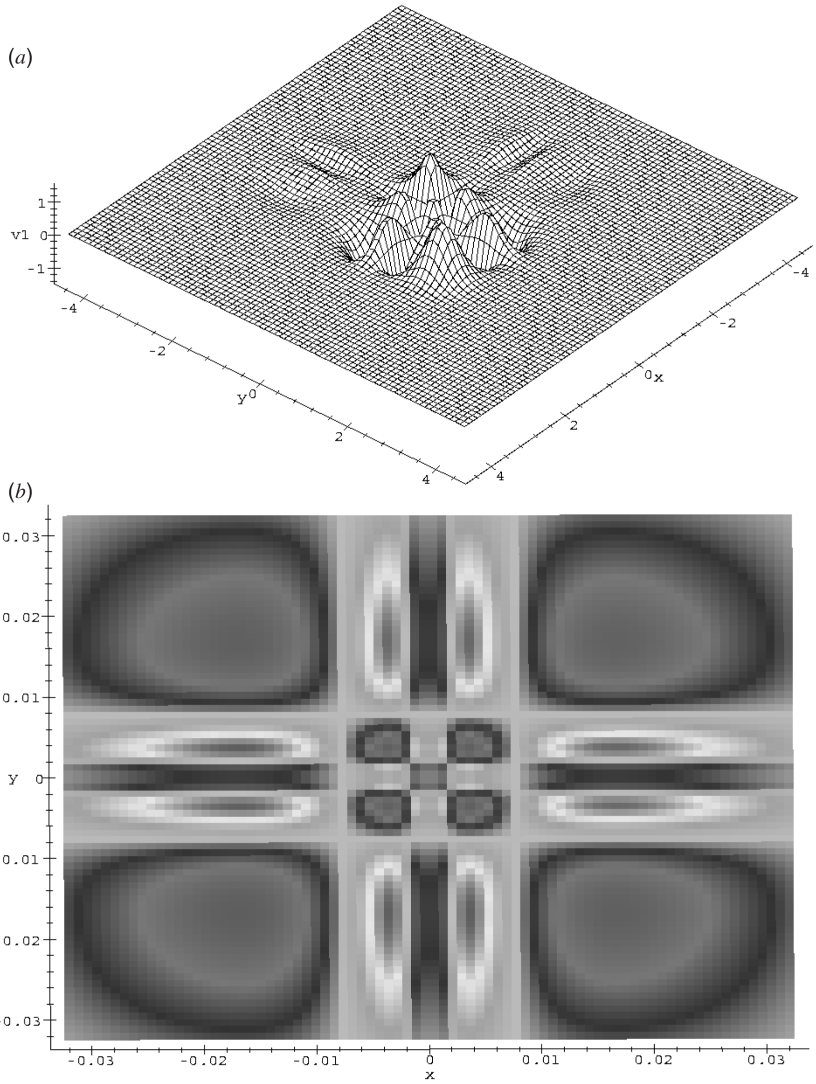

Figure 2. (*a*) Plot of the fractal dromion solution given by (13) with (17). (*b*) A density plot of the fractal dromion related to (*a*) at the region $\{x = [-0.032, 0.032], y = [-0.032, 0.032]\}$.

$\{x = [-0.000\,29, 0.000\,29], y = [-0.000\,29, 0.000\,29]\}$, $\{x = [-0.000\,06, 0.000\,06], y = [-0.000\,06, 0.000\,06]\}$ and so on, we find a totally similar structure to that presented in figure 2(*b*).

The lump solution (algebraically localized in all directions) is another type of significant localized solution in high dimensions. If we select the functions p and q of (13) appropriately, we can also find some types of fractal lump solutions. For example, selecting

$$p = \frac{|x|}{1 + x^4}(\sin(\ln(x^2)) - \cos(\ln(x^2)))^2,$$
$$q = \frac{|y|}{1 + y^4}(\sin(\ln(y^2)) - \cos(\ln(y^2)))^2, \tag{18}$$

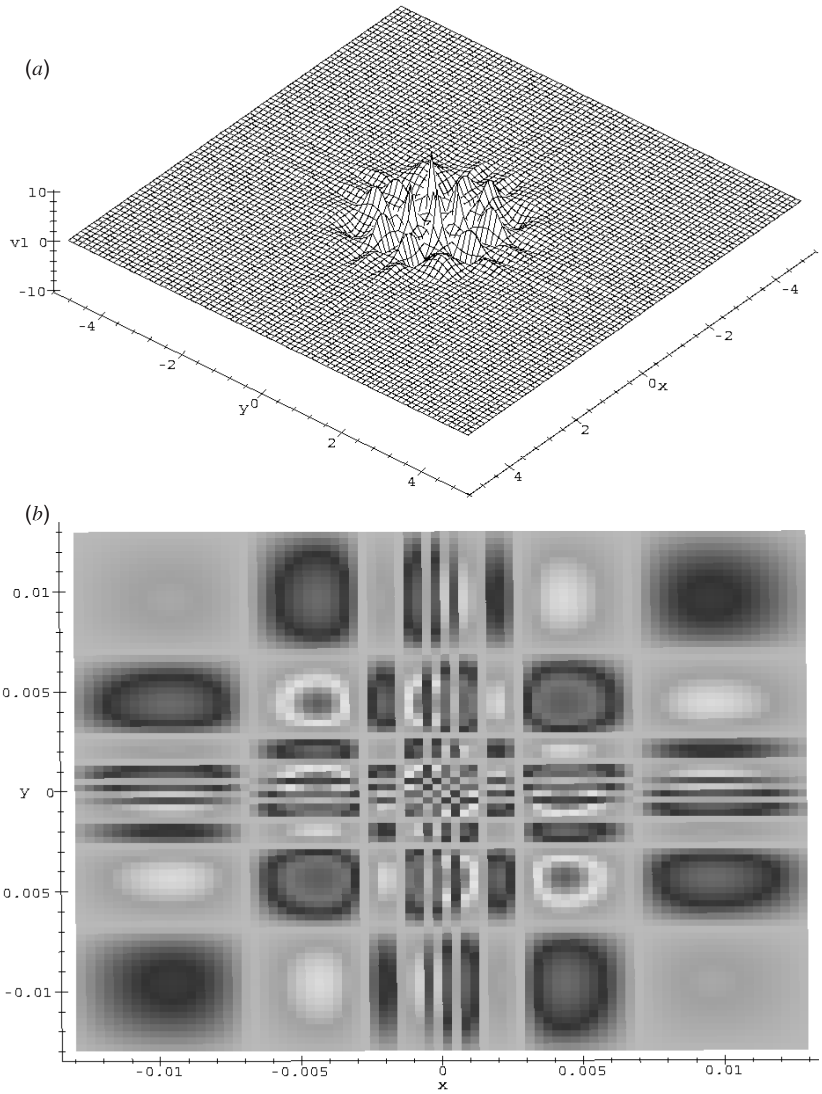

Figure 3. (*a*) Plot of the lump solution given by (13) with (18). (*b*) Density plot of the fractal lump solution related to (*a*) at the region $\{x = [-0.0128, 0.0128], y = [-0.0128, 0.0128]\}$.

and fixing the functions A_i and c_i as in (15), we obtain a lump solution with fractal structure. Figure 3(*a*) plots the lump solution (13) with (18) in three-dimensional coordinates. From figure 3(*a*), we can see that the solution is localized in all directions. Near the centre there are infinitely many peaks which are distributed in a fractal manner. To see the fractal structure of the lump solution (13) with (18) we should look at the structure more carefully. Figure 3(*b*) presents a density plot of the structure of the lump (13) with (18) at the region $\{x = [-0.14, 0.14], y = [-0.14, 0.14]\}$. More detailed studies will show us the self-similar structure of the lump. For instance, if we reduce the region of figure 3(*b*) to $\{x = [-0.03, 0.03], y = [-0.03, 0.03]\}$, $\{x = [-0.0063, 0.0063], y = [-0.0063, 0.0063]\}$, $\{x = [-0.0013, 0.0013], y = [-0.0013, 0.0013]\}$, $\{x = [-0.000\,26, 0.000\,26], y = [-0.000\,26, 0.000\,26]\}, \ldots$, we can obtain a totally similar structure to that plotted in figure 3(*b*).

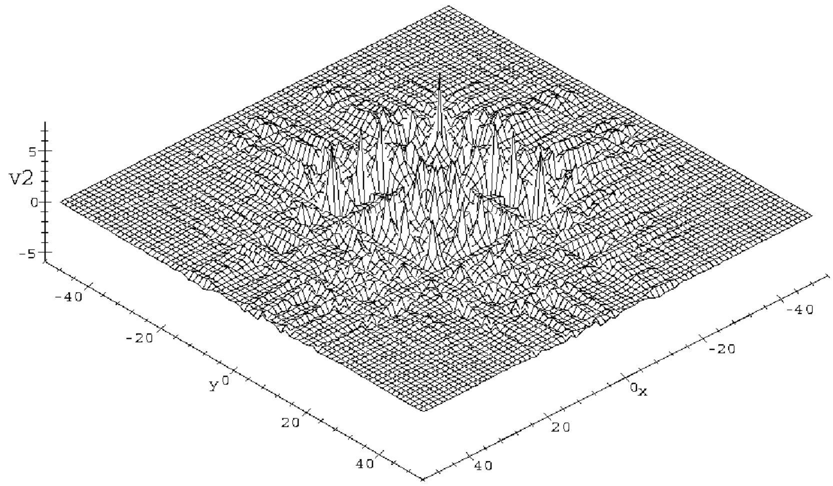

Figure 4. Plot of the stochastic lump solution given by (13) with (20).

(iii) Stochastic fractal dromions and lumps. In addition to the self-similar regular fractal dromion and lumps, the lower-dimensional stochastic fractal functions may also be used to construct high-dimensional stochastic fractal dromions and lump solutions. The most well known stochastic fractal function is the so-called Weierstrass function

$$w \equiv w(\eta) = \sum_{k=0}^{N} (\tfrac{3}{2})^{-k/2} \sin((\tfrac{3}{2})^k \eta), \qquad N \to \infty, \tag{19}$$

where the independent argument η may be a suitable function of $\{x, t\}$ and/or $\{y, t\}$, say, $\eta = x + at$ and $\eta = y$ in the functions of p and q respectively, for the following selection (20). If the Weierstrass function is included in the dromion and/or lump solutions, then we obtain the stochastic fractal dromions and lumps. Figure 4 presents a plot of a typical stochastic fractal lump solution which is determined by (13) with (19) and

$$p = w(x + at) + (x + at)^2 + 1000, \qquad q = w(y) + y^2 + 1000 \tag{20}$$

at $t = 0$. In figure 4, the vertical axis denotes the quantity $v2$ which is only a re-scaling of the field $v1$: $v2 \equiv v1/200\,000$.

Why can a dromion and/or a lump be chaotic and fractal? In [15], by means of pure numerical calculations, the authors pointed out that for the DS equation, non-chaotic dromions may be remote controlled. Actually, the chaotic and fractal behaviours of dromions are also due to the boundary conditions. Though the physical quantity $v1$ has a zero boundary condition, some other types of physical quantities or potentials such as u (11) possess nonzero chaotic and/or fractal boundary conditions. In other words, the chaotic and fractal dromions are controlled in the distance by those quantities with nonzero boundary values. In fact, every type of known dromion solution which is localized in all directions for any $(2+1)$-dimensional model can be remote controlled by some types of potentials which possess nonzero boundary conditions. It is known that for an n-dimensional N-order (Painlevé) integrable model there exist enough (N) arbitrary $(n-1)$-dimensional arbitrary characteristic functions. To find the general closed solutions with enough characteristic functions for nonlinear partial differential equations is quite difficult except for those C-integrable models [16]. For the $(2+1)$-dimensional DLWS (1), the general solution should possess five two-dimensional characteristic functions. In our special variable separation solution, only one two-dimensional and some one-dimensional arbitrary functions are included. The propagation of the exotic chaotic behaviour along the special characteristic for the field u leads to the chaotic and fractal behaviours of the dromions and lumps for the field $v1$.

Although experimental physicists have not yet found real dromion excitations in any real physical systems, we still believe that there must exist many possible applications for this special and interesting phenomenon. First of all, it can be used to produce dromion excitations in real physics. The reason for dromion excitations have not yet been found in any real physical system is that although some types of physical quantities are localized in all directions, some other types of quantities (potentials) possess nonzero boundary conditions. Hence, in order to find the dromion excitations for some kinds of physical quantities, one can take advantage of some other types of quantities by the remote control method. In this way, when the stable dromion is found, the chaotic dromion can also be immediately found by using the chaotic control signal.

Except for the chaotic dromion solutions, we also find the fractal structures for $(2 + 1)$-dimensional DLWE shown in figures 2(b) and 3(b). As is known, fractals not only belong to the realms of mathematics and computer graphics, but also exist nearly everywhere in nature, such as in tree branching, cloud structures, galaxy clustering, fern shapes, human veins, leaves, music, coastlines, fluid turbulence, crystal growth patterns and in numerous other examples. By selecting different types of lower-dimensional fractal models, one may obtain various beautiful higher dimensional fractal patterns. These beautiful pictures may be useful in costume design, architecture and so on. In addition, one can compose music by choosing appropriate forms of the arbitrary functions such as p and q. So, in the future, perhaps the most famous artists will also be the most famous physicists and/or mathematicians. Generally, chaos and fractals are the opposites to solitons in nonlinear science. Solitons are the representatives of integrable systems while chaos and fractals are on the behalf of non-integrable systems. However, in this letter, we find some chaotic and fractal structures for the lump and dromion solutions of the $(2+1)$-dimensional DWLE which is a so-called Lax and/or IST integrable equation. Naturally, the question of what on earth the integrability definition is, casts on our mind. So does the question of how to find and make use of this novel phenomena in reality.

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